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NOTE ON MULTIPLY PERFECT NUMBERS.*

BY JACOB WESTLUND.

IN the ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900/01), p. 103, Dr. D. N. Lehmer proves that no multiply perfect numbers of multiplicity 3, containing less than three distinct primes, exist. In an earlier note on Multiply Perfect Numbers† all numbers of multiplicity 3 of the form $p_1^{a_1} p_2^{a_2} p_3^{a_3}$ were determined. The object of the present note is to determine all numbers of multiplicity 3 of the form $m = p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$ where p_1, p_2, p_3, p_4 are distinct primes and $p_1 < p_2 < p_3 < p_4$.

Defining a multiply perfect number as one which is an exact divisor of the sum of all its divisors, the quotient being the multiplicity, we have‡ in the present case

$$3 = \frac{p_1 - 1/p_1^{a_1}}{p_1 - 1} \prod_{i=2}^4 \frac{p_i - 1/p_i}{p_i - 1} = \frac{p_1^{a_1+1} - 1}{p_1^{a_1}(p_1 - 1)} \prod_{i=2}^4 \frac{p_i + 1}{p_i} \quad (1)$$

and

$$3 < \prod_{i=1}^4 \frac{p_i}{p_i - 1}. \quad (2)$$

From the inequality (2) we infer that $p_1 = 2$ is the only possible value of p_1 , since the maximum value of $\prod_{i=1}^4 \frac{p_i}{p_i - 1}$ will exceed 3 only for $p_1 = 2$. Hence

$$\frac{3}{2} < \prod_{i=2}^4 \frac{p_i}{p_i - 1},$$

which gives $p_2 < 7$, i. e. the only possible values of p_2 are 3 and 5.

I. Suppose $p_2 = 3$. Then we should have from (1)

$$3 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{4}{3} \cdot \frac{p_3 + 1}{p_3} \cdot \frac{p_4 + 1}{p_4}$$

or

$$9 = \frac{2^{a_1+1} - 1}{2^{a_1-2}} \cdot \frac{p_3 + 1}{p_3} \cdot \frac{p_4 + 1}{p_4}. \quad (3)$$

* Read before the Chicago Section of the American Mathematical Society, January 3, 1902.

† Westlund, ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900/01), p. 172.

‡ Cf. Lehmer, l. c.

From this we infer that

$$\left(\frac{p_3 + 1}{p_3}\right)^2 > \frac{9 \cdot 2^{a_1-2}}{2^{a_1+1} - 1} > \frac{9}{8}$$

or
$$\frac{p_3 + 1}{p_3} > \frac{3}{4} \sqrt{2}.$$

Hence $p_3 < 17$, i. e. the only possible values of p_3 are 5, 7, 11, 13.

From (3) we see that $2^{a_1+1} - 1$ must be divisible by p_4 . Setting $2^{a_1+1} - 1 = kp_4$ we have

$$\begin{aligned} 9 \cdot 2^{a_1-2} p_3 &= k(p_3 + 1)(p_4 + 1) \\ &= (p_3 + 1)(2^{a_1+1} - 1 + k) \end{aligned} \quad (4)$$

which may be written in the following form

$$2^{a_1-2}(p_3 - 8) = (k - 1)(p_3 + 1). \quad (5)$$

Hence we must have $k > 1$ and $p_3 > 8$, and the only possible values of p_3 are 11 and 13. It is easily seen that $p_3 = 13$ does not satisfy (5). For $p_3 = 11$ we get from (5)

$$3 \cdot 2^{a_1-2} = (k - 1)12$$

or

$$2^{a_1-4} = k - 1,$$

and since $2^5 \cdot 2^{a_1-4} = kp_4 + 1$, $k = \frac{33}{32 - p_4}$. Hence $p_4 = 31$, $k = 33$, $a_1 = 9$.

The corresponding number is $m = 2^9 \cdot 3 \cdot 11 \cdot 31$ which by trial is found to be multiply perfect.

II. Suppose $p_2 = 5$. In this case we should have

$$3 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{6}{5} \cdot \frac{p_3 + 1}{p_3} \cdot \frac{p_4 + 1}{p_4} \quad (6)$$

which gives

$$\left(\frac{p_3 + 1}{p_3}\right)^2 > \frac{5 \cdot 2^{a_1-1}}{2^{a_1+1} - 1} > \frac{5}{4}$$

or

$$\frac{p_3 + 1}{p_3} > \frac{1}{2} \sqrt{5}.$$

Hence the only possible value of p_3 is 7. Setting, as in the first case, $2^{a_1+1} - 1 = kp_4$ we get from (6)

$$35 \cdot 2^{a_1-4} = k(p_4 + 1)$$

or

$$3 \cdot 2^{a_1-4} = k - 1,$$

and $k = \frac{35}{32 - 3p_4}$. Hence $p_4 < 11$ and since $p_4 > 7$ this is absurd. Therefore p_2 cannot be equal to 5, and the only multiply perfect number of multiplicity 3 of the form $m = p_1^{a_1} p_2 p_3 p_4$ is the number $2^9 \cdot 3 \cdot 11 \cdot 31$.

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